Special Functions

SMS 2308: Mathematical Methods

Nurul Farahain Mohammad

Department of Computational and Theoretical Sciences,
Kulliyyah of Science, International Islamic University Malaysia.

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Outline

Hypergeometric Equation

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0, \qquad (1)$$

where α, β and γ are fixed parameters.

- Regular singular points at $x = 0$ and $x = 1$.
- By solving this equation using Frobenius' method and factorial function, a solution for this problem is obtained.

Factorial Function

Factorial function $(t)_n$ is defined for $n \in \mathbb{Z}^+ \cup \{0\}$ by

$$\begin{aligned} (t)_n &:= t\,(t+1)\,(t+2)\cdots(t+n-1)\,, \qquad n \geqslant 1. \\ (t)_0 &:= 1, \qquad t \neq 0. \end{aligned} \tag{2}$$

Gaussian Hypergeometric Function $F(\alpha, \beta; \gamma; x)$

$$F(\alpha, \beta; \gamma; x) := 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n} x^n. \tag{3}$$

or in terms of gamma function Γ

$$F(\alpha, \beta; \gamma; x) := \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} x^n. \tag{4}$$

Gamma Function $\Gamma(x)$

$$\Gamma(x) := \int_0^\infty e^{-u} u^{x-1} du, \qquad x > 0, \tag{5}$$

$$\Gamma(x+1) = \int_0^\infty e^{-u} u^{x-1+1} du = \int_0^\infty e^{-u} u^x du,$$

$$w = u^x, \qquad dv = e^{-u} du,$$

$$dw = x u^{x-1} du, \qquad v = -e^{-u},$$

$$= wv|_0^\infty - \int_0^\infty v dw$$

$$= -e^{-u} u^x \Big|_{u=0}^{u=\infty} - \int_0^\infty -x e^{-u} u^{x-1} du,$$

$$= x \int_0^\infty e^{-u} u^{x-1} du = x\Gamma(x).$$

$$\Gamma(x+1) = x\Gamma(x). \tag{6}$$

Gamma Function $\Gamma(x)$

$$
\begin{aligned}
\Gamma(t+1) &= t\Gamma(t), \\
\Gamma(t+2) &= \Gamma([t+1]+1) = (t+1)\Gamma(t+1) = (t+1)\,t\Gamma(t), \\
\Gamma(t+3) &= \Gamma([t+2]+1) = (t+2)\Gamma(t+2) = (t+2)(t+1)\,t\Gamma(t), \\
&\vdots \\
\Gamma(t+n) &= \Gamma([t+n-1]+1) \\
&= (t+n-1)\cdots(t+2)(t+1)\,t\Gamma(t), \\
\Gamma(t+n) &= (t)_n\Gamma(t).
\end{aligned}
$$

$$
(t)_n = \frac{\Gamma(t+n)}{\Gamma(t)}. \tag{7}
$$

Bessel's Equation of order ν

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0, \tag{8}$$

where $\nu \geqslant 0$ is a fixed parameter.

- Regular singular point at $x = 0$ and no other singular points in the complex plane.
- By solving this equation using Frobenius' method, a solution for this problem is obtained.

Bessel's functions

- Bessel's function of the 1st kind of order ν :

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{9}$$

- Bessel's function of the 1st kind of order $-\nu$:

$$J_{-\nu}(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}. \tag{10}$$

- Bessel's function of the 2nd kind of order ν :

$$Y_\nu(x) := \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{sin(\nu\pi)}, \qquad \nu \notin \mathbb{Z}. \tag{11}$$

- Neumann's function :

$$N_m(x) = Y_m(x) = \lim_{\nu \to m} \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{sin(\nu\pi)}. \tag{12}$$

Useful recurrence relations

$$\frac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] = x^{\nu} J_{\nu-1}(x), \tag{13}$$

$$\begin{aligned}
\frac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] &= \frac{d}{dx}\left[x^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}\right], \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2\nu)}{n!\Gamma(1+\nu+n)\, 2^{2n+\nu}} x^{2n+2\nu-1}, \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (n+\nu)\, 2}{n!\,(\nu+n)\,\Gamma(\nu+n)\, 2^{2n+\nu}} x^{2n+2\nu-1}, \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n)\, 2^{2n+\nu-1}} x^{(2n+\nu-1)+\nu}, \\
&= x^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} = x^{\nu} J_{\nu}(x).
\end{aligned}$$

$$\frac{d}{dx}\left[x^{-\nu} J_{\nu}(x)\right] = -x^{\nu} J_{\nu+1}(x), \tag{14}$$

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x) - J_{\nu-1}(x), \tag{15}$$

$$J_{\nu+1}(x) = J_{\nu-1}(x) - 2J'_{\nu}(x). \tag{16}$$

Approximation of Bessel functions

- Approximation of Bessel functions for large arguments $x \gg 1$:

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),$$
$$Y_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right). \tag{17}$$

- Approximation of Bessel functions for small arguments $0 < x \ll 1$:

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(1+\nu)},$$
$$Y_0(x) \approx \frac{2\ln x}{\pi}, \qquad Y_{\nu>0}(x) \approx -\frac{\Gamma(\nu)\, 2^\nu}{\pi x^\nu}. \tag{18}$$

Legendre's Equation

$$\left(1-x^2\right)y''-2xy'+n\left(n+1\right)y=0, \tag{19}$$

where n is a fixed parameter.

- Regular singular point at $x=1$.
- By solving this equation using power series, a solution for this problem is obtained.

Legendre polynomials / Spherical polynomials

$$P_n(x) := 1 + \sum_{k=1}^{\infty} \frac{(-n)_k (n+1)_k}{k!(1)_k} \left(\frac{1-x}{2}\right)^k. \tag{20}$$

If we expand about $x = 0$, then P_n takes the form

$$P_n(x) = 2^{-n} \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m)!}{(n-m)!m!(n-2m)!} x^{n-2m}, \tag{21}$$

where $[n/2]$ is the greatest integer less than or equal to $n/2$.

Orthogonality condition

The legendre polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} P_m(x)\, P_n(x) dx = 0, \qquad n \neq m. \tag{22}$$

Orthogonality condition

$$\left(1-x^2\right)y''-2xy'+n\left(n+1\right)y=0,$$
$$\left[\left(1-x^2\right)y'\right]'+n\left(n+1\right)y=0,$$
$$\left[\left(1-x^2\right)P_m'\right]'+n\left(n+1\right)P_m=0,$$
$$\left[\left(1-x^2\right)P_n'\right]'+m\left(m+1\right)P_n=0,$$
$$P_n\left[\left(1-x^2\right)P_m'\right]'+n\left(n+1\right)P_mP_n=0,$$
$$P_m\left[\left(1-x^2\right)P_n'\right]'+m\left(m+1\right)P_mP_n=0,$$

$$P_n\left[\left(1-x^2\right)P_m'\right]'+n\left(n+1\right)P_mP_n-P_m\left[\left(1-x^2\right)P_n'\right]'-m\left(m+1\right)P_mP_n=0,$$
$$\left[n\left(n+1\right)-m\left(m+1\right)\right]P_mP_n=\left[\left(1-x^2\right)\left(P_nP_m'-P_n'P_m\right)\right]',$$
$$\int_{x=-1}^{x=1}\left[n\left(n+1\right)-m\left(m+1\right)\right]P_mP_ndx=\int_{x=-1}^{x=1}\left[\left(1-x^2\right)\left(P_nP_m'-P_n'P_m\right)\right]'dx,$$
$$\left[n\left(n+1\right)-m\left(m+1\right)\right]\int_{x=-1}^{x=1}P_mP_ndx=\left[\left(1-x^2\right)\left(P_nP_m'-P_n'P_m\right)\right]\Big|_{x=-1}^{x=1},$$
$$\int_{x=-1}^{x=1}P_mP_ndx=0.$$

Recurrence formula and Rodrigues' formula

Legendre polynomials also satisfy the recurrence formula

$$(n+1)\,P_{n+1}(x) = (2n+1)\,xP_n(x) - nP_{n-1}(x)\,. \qquad (23)$$

and Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left\{\left(x^2-1\right)^n\right\}. \qquad (24)$$

Generating function for $P_n(x)$

Function

$$\left(1-2xz+z^2\right)^{-1/2} \tag{25}$$

is used as generating function for $P_n(x)$ such that

$$\left(1-2xz+z^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n\left(x\right) z^n, \qquad |z|<1, \qquad |x|<1.$$