Special Functions

SMS 2308: Mathematical Methods

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Hypergeometric Equation

\[ x (1 - x) y'' + [\gamma - (\alpha + \beta + 1) x] y' - \alpha \beta y = 0, \quad (1) \]

where \( \alpha, \beta \) and \( \gamma \) are fixed parameters.

- Regular singular points at \( x = 0 \) and \( x = 1 \).
- By solving this equation using Frobenius’ method and factorial function, a solution for this problem is obtained.
Factorial Function

Factorial function \((t)_n\) is defined for \(n \in \mathbb{Z}^+ \cup \{0\}\) by

\[
(t)_n := t(t + 1)(t + 2) \cdots (t + n - 1), \quad n \geq 1.
\]

\[
(t)_0 := 1, \quad t \neq 0.
\]
Gaussian Hypergeometric Function \( F(\alpha, \beta; \gamma; x) \)

\[
F (\alpha, \beta; \gamma; x) := 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n. \tag{3}
\]

or in terms of gamma function \( \Gamma \)

\[
F (\alpha, \beta; \gamma; x) := \frac{\Gamma (\gamma)}{\Gamma (\alpha) \Gamma (\beta)} \sum_{n=0}^{\infty} \frac{\Gamma (\alpha + n) \Gamma (\beta + n)}{n! \Gamma (\gamma + n)} x^n. \tag{4}
\]
**Gamma Function** $\Gamma(x)$

$$
\Gamma(x) := \int_{0}^{\infty} e^{-u} u^{x-1} \, du, \quad x > 0, \quad (5)
$$

$$
\Gamma(x + 1) = \int_{0}^{\infty} e^{-u} u^{x+1} \, du = \int_{0}^{\infty} e^{-u} u^{x} \, du,
$$

where

\[ w = u^x, \quad dv = e^{-u} \, du, \]
\[ dw = xu^{x-1} \, du, \quad v = -e^{-u}, \]

\[ wv|_0^\infty = -e^{-u}u^x|_{u=0}^{u=\infty} - \int_{0}^{\infty} -xe^{-u}u^{x-1} \, du, \]

\[ = x \int_{0}^{\infty} e^{-u} u^{x-1} \, du = x\Gamma(x). \]

$$
\Gamma(x + 1) = x\Gamma(x). \quad (6)
$$
Gamma Function $\Gamma(x)$

\[
\begin{align*}
\Gamma(t + 1) &= t\Gamma(t), \\
\Gamma(t + 2) &= \Gamma([t + 1] + 1) = (t + 1)\Gamma(t + 1) = (t + 1) t\Gamma(t), \\
\Gamma(t + 3) &= \Gamma([t + 2] + 1) = (t + 2)\Gamma(t + 2) = (t + 2)(t + 1) t\Gamma(t), \\
&\quad \vdots \\
\Gamma(t + n) &= \Gamma([t + n - 1] + 1) \\
&= (t + n - 1)\cdots(t + 2)(t + 1) t\Gamma(t), \\
\Gamma(t + n) &= (t)n\Gamma(t).
\end{align*}
\]

\[
(t)_n = \frac{\Gamma(t + n)}{\Gamma(t)}. \tag{7}
\]
Bessel’s Equation of order $\nu$

\[ x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad (8) \]

where $\nu \geq 0$ is a fixed parameter.

- Regular singular point at $x = 0$ and no other singular points in the complex plane.
- By solving this equation using Frobenius’ method, a solution for this problem is obtained.
Bessel’s functions

- Bessel’s function of the 1st kind of order $\nu$:

$$J_{\nu}(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu}.$$  \hspace{1cm} (9)

- Bessel’s function of the 1st kind of order $-\nu$:

$$J_{-\nu}(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 - \nu + n)} \left(\frac{x}{2}\right)^{2n-\nu}.$$  \hspace{1cm} (10)

- Bessel’s function of the 2nd kind of order $\nu$:

$$Y_{\nu}(x) := \frac{\cos(\nu \pi) J_{\nu}(x) - J_{-\nu}(x)}{\sin(\nu \pi)}, \quad \nu \notin \mathbb{Z}. \hspace{1cm} (11)$$

- Neumann’s function:

$$N_m(x) = Y_m(x) = \lim_{\nu \to m} \frac{\cos(\nu \pi) J_{\nu}(x) - J_{-\nu}(x)}{\sin(\nu \pi)}.$$  \hspace{1cm} (12)
Useful recurrence relations

\[
\frac{d}{dx} \left[ x^\nu J_\nu (x) \right] = x^\nu J_{\nu - 1} (x), \quad (13)
\]

\[
\frac{d}{dx} \left[ x^\nu J_\nu (x) \right] = \frac{d}{dx} \left[ x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma (1 + \nu + n)} \left( \frac{x}{2} \right)^{2n+\nu} \right],
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 2\nu)}{n! \Gamma (1 + \nu + n) 2^{2n+\nu}} x^{2n+2\nu - 1},
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n (n + \nu) 2n}{n! (\nu + n) \Gamma (\nu + n) 2^{2n+\nu}} x^{2n+2\nu - 1},
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma (\nu + n) 2^{2n+\nu - 1}} x^{(2n+\nu - 1)+\nu},
\]

\[
= x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma (\nu + n)} \left( \frac{x}{2} \right)^{2n+\nu - 1} = x^\nu J_\nu (x).
\]

\[
\frac{d}{dx} \left[ x^{-\nu} J_\nu (x) \right] = -x^\nu J_{\nu + 1} (x), \quad (14)
\]

\[
J_{\nu + 1} (x) = \frac{2\nu}{x} J_\nu (x) - J_{\nu - 1} (x), \quad (15)
\]

\[
J_{\nu + 1} (x) = J_{\nu - 1} (x) - 2J'_\nu (x). \quad (16)
\]
Approximation of Bessel functions

▶ Approximation of Bessel functions for large arguments $x \gg 1$:

\[ J_\nu (x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right), \]
\[ Y_\nu (x) \approx \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right). \]  \hfill (17)

▶ Approximation of Bessel functions for small arguments $0 < x \ll 1$:

\[ J_\nu (x) \approx \frac{x^\nu}{2^\nu \Gamma (1 + \nu)}, \]
\[ Y_0 (x) \approx \frac{2 \ln x}{\pi}, \quad Y_{\nu > 0} (x) \approx -\frac{\Gamma (\nu) 2^\nu}{\pi x^\nu}. \]  \hfill (18)
Legendre’s Equation

\[(1 - x^2) y'' - 2xy' + n(n + 1)y = 0, \quad (19)\]

where \(n\) is a fixed parameter.

- Regular singular point at \(x = 1\).
- By solving this equation using power series, a solution for this problem is obtained.
Legendre polynomials / Spherical polynomials

\[ P_n(x) := 1 + \sum_{k=1}^{\infty} \frac{(-n)_k(n+1)_k}{k!(1)_k} \left( \frac{1-x}{2} \right)^k. \] (20)

If we expand about \( x = 0 \), then \( P_n \) takes the form

\[ P_n(x) = 2^{-n} \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m)!}{(n-m)!m!(n-2m)!} x^{n-2m}, \] (21)

where \([n/2]\) is the greatest integer less than or equal to \( n/2 \).
Orthogonality condition

The Legendre polynomials satisfy the orthogonality condition

\[ \int_{-1}^{1} P_m(x) P_n(x) \, dx = 0, \quad n \neq m. \] (22)
Orthogonality condition

\[(1 - x^2) y''' - 2xy' + n (n + 1) y = 0,\]
\[\left[(1 - x^2) y'\right]' + n (n + 1) y = 0,\]
\[\left[(1 - x^2) P'_m\right]' + n (n + 1) P_m = 0,\]
\[\left[(1 - x^2) P'_n\right]' + m (m + 1) P_n = 0,\]
\[P_n \left[(1 - x^2) P'_m\right] + n (n + 1) P_m P_n = 0,\]
\[P_m \left[(1 - x^2) P'_n\right] + m (m + 1) P_m P_n = 0,\]
\[P_n \left[(1 - x^2) P'_m\right] + n (n + 1) P_m P_n - P_m \left[(1 - x^2) P'_n\right] - m (m + 1) P_m P_n = 0,\]
\[n (n + 1) - m (m + 1) \int_{-1}^{1} P_m P_n dx = \int_{-1}^{1} \left[(1 - x^2) \left(P_n P'_m - P'_n P_m\right)\right] dx,\]
\[\int_{-1}^{1} \left[n (n + 1) - m (m + 1)\right] P_m P_n dx = \int_{-1}^{1} \left[(1 - x^2) \left(P_n P'_m - P'_n P_m\right)\right] dx,\]
\[\int_{-1}^{1} P_m P_n dx = \left[(1 - x^2) \left(P_n P'_m - P'_n P_m\right)\right]_{x=-1}^{x=1},\]
\[\int_{-1}^{1} P_m P_n dx = 0.\]
Recurrence formula and Rodrigues’ formula

Legendre polynomials also satisfy the recurrence formula

\[(n + 1) P_{n+1} (x) = (2n + 1) x P_n (x) - n P_{n-1} (x). \quad (23)\]

and Rodrigues’ formula

\[P_n (x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left\{ (x^2 - 1)^n \right\}. \quad (24)\]
Generating function for $P_n(x)$

Function

$$(1 - 2xz + z^2)^{-1/2}$$

is used as generating function for $P_n(x)$ such that

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) z^n, \quad |z| < 1, \quad |x| < 1.$$