

Cauchy - Euler equation:

$$ax^2 y''(x) + bx y'(x) + cy(x) = 0, \quad x > 0, \quad \text{--- (1)}$$

$\Rightarrow$  divide by  $ax^2$ ,

$$y''(x) + \underbrace{\frac{bx}{ax^2} y'(x)}_{p(x)} + \underbrace{\frac{c}{ax^2} y(x)}_{q(x)} = 0,$$

$$p(x) = \frac{p_0}{x}, \quad q(x) = \frac{q_0}{x^2}, \quad q_0 = \frac{c}{a}.$$

$$p_0 = \frac{b}{a}.$$

$$\Rightarrow y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad \text{--- (2)}$$

$\Rightarrow$  Substitute  $w(r, x) = x^r$  for  $y$ .

$$w' = rx^{r-1}.$$

$$w'' = r(r-1)x^{r-2}.$$

$$r(r-1)x^{r-2} + p(x)rx^{r-1} + q(x)x^r = 0,$$

$$\Rightarrow r(r-1)x^{r-2} + \underbrace{p(x)rx^{r-2}}_{p_0 x} + \underbrace{q(x)x^r}_{q_0 x^2} = 0,$$

$$p_0 = x p(x), \quad q_0 = x^2 q(x)$$

$$\Rightarrow [r(r-1) + p_0 r + q_0] x^{r-2} = 0.$$

$$\text{Let } x^{r-2} \neq 0, \Rightarrow r(r-1) + p_0 r + q_0 = 0, \quad \text{--- (3)}$$

Thus, if  $r_i$  is a root of (3), then  $w(r_i, x) = x^{r_i}$  is a solution to equations (1) and (2).

Assume (2) contains  $x p(x)$  and  $x^2 q(x)$  of analytic functions.

In some open interval about  $x=0$ ,

$$x p(x) = p_0 + p_1 x + p_2 x^2 + \dots = \sum_{n=0}^{\infty} p_n x^n, \quad \text{--- (4)}$$

$$x^2 q(x) = q_0 + q_1 x + q_2 x^2 + \dots = \sum_{n=0}^{\infty} q_n x^n. \quad \text{--- (5)}$$

$\Rightarrow$  Find limit ~~about~~  $x=0$ ,

$$\lim_{x \rightarrow 0} x p(x) = p_0, \quad \lim_{x \rightarrow 0} x^2 q(x) = q_0. \quad \text{--- (6)}$$

Hence, near  $x=0$ ,  $x p(x) \approx p_0$ , and  $x^2 q(x) \approx q_0$ .

$\Rightarrow \therefore$  Solutions to (2) will behave (for  $x$  near 0) like the solutions to the Cauchy-Euler equation

$$\underline{x^2 y'' + p_0 x y' + q_0 y = 0}.$$

Definition: Regular singular point.

A singular point  $x_0$  of  $y''(x) + p(x)y'(x) + q(x)y(x) = 0$  is said to be a regular singular point if both  $(x-x_0)p(x)$  and  $(x-x_0)^2q(x)$  are analytical at  $x_0$ .

Otherwise  $x_0$  is called an irregular singular point.

Ex. 1 : Classify the singular points of the equation

$$(x^2 - 1)^2 y''(x) + (x+1) y'(x) - y(x) = 0. \quad \text{--- } \textcircled{8}$$

#1 Transform into general form.

$$y''(x) + \underbrace{\frac{(x+1)}{(x^2-1)^2}}_{p(x)} y'(x) - \underbrace{\frac{1}{(x^2-1)^2} y(x)}_{q(x)} = 0.$$

$$p(x) = \frac{x+1}{(x^2-1)^2} = \frac{x+1}{(x+1)(x-1)^2} = \frac{1}{(x+1)(x-1)^2}.$$

$$q(x) = \frac{-1}{(x^2-1)^2} = \frac{-1}{(x+1)^2(x-1)^2}.$$

#2 Take denominator equals to zero. Why?

$$(x+1)(x-1)^2 \neq 0, \Rightarrow x \neq -1, 1.$$

$\frac{1}{0} = \infty$ :  
indeterminate form.

Thus,  $x = \pm 1$  are singular points.

#3 Check analytic or not.

For the singularity at 1,

$$(x-1)p(x) = \frac{1}{(x+1)(x-1)},$$

which is not analytic at  $x=1$ .

$\therefore x=1$  is an irregular singular point.

For singularity at -1,

$$(x+1)g(x) = \frac{1}{(x-1)^2},$$

$$(x+1)^2 g(x) = \frac{-1}{(x-1)^2},$$

both analytic at -1. Therefore  $x = -1$  is a regular singular point. \*

### ! Definition: Analytic Function.

A function  $f$  is said to be analytic at  $x_0$  if in an open interval about  $x_0$ ,

this function is the sum of a power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$
 that has positive radius of convergence.

!  $f$  is differentiable in a neighborhood of  $x_0$

↓

$f$  is analytic at  $x_0$ .

! Any power series ~~that~~ regardless of how it is derived - that converges in some neighborhood of  $x_0$  to a function has to be the Taylor series of that function.

## Frobenius

Assume  $x=0$  is a regular singular point

$$\text{for } y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

so that  $p(x)$  and  $q(x)$  satisfy  $\oplus$  &  $\ominus$ ,

$$p(x) = \sum_{n=0}^{\infty} p_n x^{n-1}, \quad q(x) = \sum_{n=0}^{\infty} q_n x^{n-2}. \quad \text{--- (a)}$$

Frobenius' idea:

since Cauchy-Euler equations have solutions of the form  $x^r$ ,

then for regular singular point  $x=0$ ,

there should be solutions of the form

$x^r$  times an analytic function. Hence,

$$w(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x > 0. \quad \text{--- (b)}$$

$a_0$  assumed as is the first nonzero coefficient,  
so we are left with determining  $r$  and  
the coefficients  $a_n, n \geq 1$ .

Differentiating  $w(r, x)$  with respect to  $x$ , we have

$$w'(r, x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \quad \text{--- (c)}$$

$$w''(r, x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}. \quad \text{--- (d)}$$

(a), (b), (c), (d)  $\Rightarrow$  ⑦.

$$\begin{aligned} & \left\{ \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right. \\ & + \left( \sum_{n=0}^{\infty} p_n x^{n-1} \right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ & + \left. \left( \sum_{n=0}^{\infty} q_n x^{n-2} \right) \sum_{n=0}^{\infty} a_n x^{n+r} \right\} = 0. \end{aligned} \quad \text{--- (e)}$$

Use Cauchy product to perform series multiplications and then group like powers of  $x$ , starting with the lowest power,  $x^{r-2}$ .  
 $(n=0 \text{ and } 1)$ :

$$\Rightarrow r(r-1)a_0x^{r-2} + (r+1)r a_1 x^{r-1} + \dots \\ + p_0 x^{-1} r a_0 x^{r-1} + p_1 (r+1) a_1 x^r + \dots \\ + q_0 x^{-2} a_0 x^r + q_1 x^{-1} a_1 x^{r+1} + \dots \\ + p_0 x^{-1} r a_0 x^{r-1} + p_1 x^{-1} (r+1) a_1 x^r + \dots \\ + p_1 r a_0 x^{r-1} + p_1 (r+1) a_1 x^r + \dots \\ + q_0 x^{-2} a_0 x^r + q_0 x^{-2} a_1 x^{r+1} + \dots \\ + q_1 x^{-1} a_0 x^r + q_1 x^{-1} a_1 x^{r+1} + \dots = 0,$$

$$\Rightarrow [r(r-1) + p_0 r + q_0] a_0 x^{r-2} \\ + [(r+1)r a_1 + p_0(r+1) a_1 + p_1 r a_0 + q_0 a_1 + q_1 a_0] x^{r-1} + \dots = 0, \quad \textcircled{14}$$

For the expansion on the LHS of  $\textcircled{14}$ . To sum to zero, each coefficient must be zero.

Considering the first term  $x^{r-2}$ , we find

$$[r(r-1) + p_0 r + q_0] a_0 = 0. \quad \textcircled{15}$$

Since we assume  $a_0 \neq 0$ ,

$$\Rightarrow r(r-1) + p_0 r + q_0 = 0. \quad \text{(indicial equation)}$$