

SPECIAL FUNCTIONS

Hypergeometric Equation

Linear 2nd order differential equation

$$x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0, \quad \text{--- (1)}$$

where $\alpha, \beta,$ and γ are parameters.

○ singular points at $x=0$ and 1 . Both regular

$$y'' + \frac{[\gamma - (\alpha + \beta + 1)x]}{x(1-x)} y' - \frac{\alpha\beta}{x(1-x)} y = 0,$$

$$p(x) = \frac{[\gamma - (\alpha + \beta + 1)x]}{x(1-x)}$$

$$q(x) = -\frac{\alpha\beta}{x(1-x)}$$

Take denominator = 0,

$$x(1-x) = 0,$$

$x=0, x=1$ are singular points

○ For $x=0$,

$$(x \neq 0) \frac{[\gamma - (\alpha + \beta + 1)x]}{x(1-x)} = \frac{[\gamma - (\alpha + \beta + 1)x]}{1-x}$$

$$(x=0)^2 \frac{-\alpha\beta}{x(1-x)} = -\frac{\alpha\beta x}{1-x}$$

∴ analytic

∴ regular singular point

☞ For $x=1$,

$$(x-1) \frac{[\gamma - (\alpha + \beta + 1)x]}{-x(x-1)} = \alpha + \beta + 1 - \frac{\gamma}{x}$$

$$(x-1)^2 \frac{-\alpha\beta}{-x(x-1)} = \frac{\alpha\beta(x-1)}{x}$$

∴ analytic

∴ regular singular point.

⊙ Indicial equation associated with $x=0$.

$$p_0 = \lim_{x \rightarrow 0} \frac{\gamma - (\alpha + \beta + 1)x}{1-x} = \gamma$$

$$q_0 = \lim_{x \rightarrow 0} \frac{-\alpha\beta x}{1-x} = 0$$

$$\Rightarrow r(r-1) + \gamma r + 0 = 0,$$

$$r(r-1 + \gamma) = 0,$$

$$r = 0 \quad \text{or} \quad r - 1 + \gamma = 0,$$

$$\Rightarrow r = 1 - \gamma.$$

Assume $\gamma \notin \mathbb{Z}$, use $r=0$ to obtain solution of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n,$$

————— (2)

$$\Rightarrow y_1'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1},$$

$$\Rightarrow y_1''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2},$$

Substitute (2) and its derivatives in (1),

$$x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0,$$

$$\Rightarrow x(1-x) \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

$$+ [\gamma - (\alpha + \beta + 1)x] \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$- \alpha\beta \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$+ \sum_{n=0}^{\infty} \gamma n a_n x^{n-1} - \sum_{n=0}^{\infty} (\alpha + \beta + 1)n a_n x^n$$

$$- \sum_{n=0}^{\infty} \alpha\beta a_n x^n = 0,$$

$$\Rightarrow \sum_{n=0}^{\infty} [n(n-1 + \gamma)a_n] x^{n-1} - \sum_{n=0}^{\infty} [n(n-1 + \alpha + \beta + 1) + \alpha\beta] a_n x^n = 0,$$

$$\Rightarrow \sum_{n=0}^{\infty} [n(n-1 + \gamma)a_n] x^{n-1} - \sum_{n=0}^{\infty} [n(n + \alpha + \beta) + \alpha\beta] a_n x^n = 0,$$

\Rightarrow ∇ shift indices so that the power of x will be $n-1$, ∇
 Replace with dummy variable k which $k = n+1$,

$$\sum_{k=1}^{\infty} [(k-1)(k-1 + \alpha + \beta) + \alpha\beta] a_{k-1} x^{k-1},$$

$$k^2 - k + k\alpha + k\beta - k + 1 - \alpha - \beta + \alpha\beta$$

$$k^2 + k\alpha - k + k\beta - k + (1 - \alpha)(1 - \beta)$$

$$k^2 + k(\alpha - 1 + \beta - 1) + (\alpha - 1)(\beta - 1)$$

$$(k + \alpha - 1)(k + \beta - 1)$$

$$\sum_{k=1}^{\infty} [(k + \alpha - 1)(k + \beta - 1)] a_{k-1} x^{k-1}$$

$$\Rightarrow \cancel{0(0-1+\gamma)a_0 x^{-1}} + \sum_{n=1}^{\infty} n(n-1+\gamma)a_n x^{n-1} + \sum_{n=1}^{\infty} (n+\alpha-1)(n+\beta-1)a_{n-1} x^{n-1} = 0,$$

$$\Rightarrow \sum_{n=1}^{\infty} [n(n-1+\gamma)a_n + (n+\alpha-1)(n+\beta-1)a_{n-1}] x^{n-1} = 0, \quad \text{--- (2)}$$

Set the series coefficients equal to zero yields the recurrence relation

$$n(n-1+\gamma) a_n - (n+\alpha-1)(n+\beta-1) a_{n-1} = 0, \quad n \geq 1. \quad \text{--- (4)}$$

$$\Rightarrow a_n = \frac{(n+\alpha-1)(n+\beta-1)}{n(n-1+\gamma)} a_{n-1}. \quad \text{--- (5)}$$

$$\Rightarrow a_n = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n! \gamma(\gamma+1)\dots(\gamma+n-1)} a_0, \quad n \geq 1. \quad \text{--- (6)}$$

If employ the factorial function $(t)_n$, which is defined for nonnegative integers n by

$$(t)_n := t(t+1)(t+2)\dots(t+n-1), \quad n \geq 1.$$

$$(t)_0 := 1, \quad t \neq 0,$$

then a_n can be expressed as

$$a_n = \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} a_0, \quad n \geq 1.$$

Take $a_0 = 1$,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n, \\ &= a_0 + \sum_{n=1}^{\infty} a_n x^n, \\ &= 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} x^n. \end{aligned}$$

Thus, the solution is called a Gaussian hypergeometric function. General form of it is

$$F(\alpha, \beta; \gamma; x) := 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} x^n.$$

⊙ Generalisations of the geometric series

Use the other root, $r=1-\gamma$, to seek a second linearly independent solution of the form

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+1-\gamma} \quad \text{--- (5)}$$

$$y_2'(x) = \sum_{n=0}^{\infty} (n+1-\gamma) b_n x^{n-\gamma}$$

$$y_2''(x) = \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma) b_n x^{n-\gamma-1}$$

Substitute (5) and its derivatives in (1).

$$\begin{aligned} & x(1-x) \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma) b_n x^{n-\gamma-1} \\ & + [\gamma - (\alpha + \beta + 1)x] \sum_{n=0}^{\infty} (n+1-\gamma) b_n x^{n-\gamma} \\ & - \alpha\beta \sum_{n=0}^{\infty} b_n x^{n+1-\gamma} = 0, \end{aligned}$$

$$\begin{aligned} \Rightarrow & \left(\sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma) b_n x^{n-\gamma} - \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma) b_n x^{n-\gamma+1} \right) \\ & + \left(\sum_{n=0}^{\infty} \gamma(n+1-\gamma) b_n x^{n-\gamma} - \sum_{n=0}^{\infty} (\alpha + \beta + 1)(n+1-\gamma) b_n x^{n-\gamma+1} \right) \\ & - \sum_{n=0}^{\infty} \alpha\beta b_n x^{n-\gamma+1} \end{aligned}$$

$$\begin{aligned} \Rightarrow & \sum_{n=0}^{\infty} (n+1-\gamma) n b_n x^{n-\gamma} \\ & - \sum_{n=0}^{\infty} [(n+1-\gamma)(n-\gamma + \alpha + \beta + 1) + \alpha\beta] b_n x^{n-\gamma+1} = 0, \end{aligned}$$

\Rightarrow Shift index, $k=n+1, \Rightarrow n=k-1$, for the last term

$$\sum_{k=1}^{\infty} [(k-\gamma)(k-\gamma + \alpha + \beta) + \alpha\beta] b_{k-1} x^{k-\gamma}$$

$$= \sum_{k=1}^{\infty} [k^2 - k\gamma + k\alpha + k\beta - k\gamma + \gamma^2 + \gamma\alpha - \gamma\beta + \alpha\beta] b_{k-1} x^{k-\gamma},$$

$$= \sum_{k=1}^{\infty} [k^2 + k(-2\gamma + \alpha + \beta) + (\gamma - \alpha)(\gamma - \beta)] b_{k-1} x^{k-\gamma},$$

$$= \sum_{k=1}^{\infty} [k^2 + k(\alpha - \gamma + \beta - \gamma) + (\alpha - \gamma)(\beta - \gamma)] b_{k-1} x^{k-\gamma}$$

$$= \sum_{k=1}^{\infty} [(k + \alpha - \gamma)(k + \beta - \gamma)] b_{k-1} x^{k-\gamma}$$

change the dummy variable k into n ,

$$= \sum_{n=1}^{\infty} [(n + \alpha - \gamma)(n + \beta - \gamma)] b_{n-1} x^{n-\gamma}$$

$$\Rightarrow (0+1-\gamma)0 b_0 x^{-\gamma} + \sum_{n=1}^{\infty} (n+1-\gamma)n b_n x^{n-\gamma} - \sum_{n=1}^{\infty} [(n+\alpha-\gamma)(n+\beta-\gamma)] b_{n-1} x^{n-\gamma} = 0,$$

$$\Rightarrow \sum_{n=1}^{\infty} [(n+1-\gamma)n b_n - (n+\alpha-\gamma)(n+\beta-\gamma) b_{n-1}] x^{n-\gamma} = 0,$$

since $x^{n-\gamma} \neq 0$,

Thus we have recurrence relation

$$(n+1-\gamma)n b_n - (n+\alpha-\gamma)(n+\beta-\gamma) b_{n-1} = 0,$$

$$(n+1-\gamma)n b_n = (n+\alpha-\gamma)(n+\beta-\gamma) b_{n-1} = l,$$

$$b_n = \frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{n(n+1-\gamma)} b_{n-1}$$

$$b_n = \frac{(1+\alpha-\gamma)(2+\alpha-\gamma) \dots (n+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma) \dots (n+\beta-\gamma)}{n!(2-\gamma)(3-\gamma) \dots (n+1-\gamma)} b_0$$

\star factorial function

$$(t)_n := t(t+1)(t+2) \dots (t+n-1),$$

$$(t+1)_n := (t+1)(t+2)(t+3) \dots (t+n),$$

$$(t+1-\gamma)_n := (t+1-\gamma)(t+2-\gamma)(t+3-\gamma) \dots (t+n-\gamma),$$

$$(2-t)_n := (2-t)(3-t) \dots (n+1-t).$$

$$\Rightarrow b_n = \frac{(1+\alpha-\gamma)(2+\alpha-\gamma) \dots (n+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma) \dots (n+\beta-\gamma)}{n!(2-\gamma)_n} b_0$$

$$b_n = \frac{(1+\alpha-\gamma)_n (1+\beta-\gamma)_n}{n!(2-\gamma)_n} b_0$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+1-\gamma}, \quad \text{--- (15)}$$

$$= b_0 x^{1-\gamma} + \sum_{n=1}^{\infty} b_n x^{n+1-\gamma},$$

assume $b_0 = 1$,

$$= x^{1-\gamma} + \sum_{n=1}^{\infty} \frac{(\alpha+1-\gamma)_n (\beta+1-\gamma)_n}{n! (2-\gamma)_n} x^{n+1-\gamma} \quad \text{--- (16)}$$

Factor out $x^{1-\gamma}$,

$$y_2(x) = x^{1-\gamma} + x^{1-\gamma} \sum_{n=1}^{\infty} \frac{(\alpha+1-\gamma)_n (\beta+1-\gamma)_n}{n! (2-\gamma)_n} x^n,$$

$$= x^{1-\gamma} \left(1 + \sum_{n=1}^{\infty} \frac{(\alpha+1-\gamma)_n (\beta+1-\gamma)_n}{n! (2-\gamma)_n} x^n \right),$$

$$= x^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x). \quad \text{--- (17)}$$

Definition of Gamma Function:

$$\Gamma(x) := \int_0^{\infty} e^{-u} u^{x-1} du, \quad x > 0. \quad \text{--- (18)}$$

It is shown in section 7.6,

$$\Gamma(x+1) = x \Gamma(x), \quad x > 0. \quad \text{--- (19)}$$

Repeated use of relation (19) yields

$$(t)_n = \frac{\Gamma(t+n)}{\Gamma(t)}, \quad t > 0, \quad n \in \mathbb{Z}^+ \cup \{0\}. \quad \text{--- (20)}$$

(20) \Rightarrow (10):

$$F(\alpha, \beta; \gamma; x) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} x^n = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} x^n,$$

\Rightarrow

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{\Gamma(\beta+n)}{\Gamma(\beta)}}{n! \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}} x^n,$$

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n) \Gamma(\beta+n)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\gamma)}{n! \Gamma(\gamma+n)} x^n,$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n) \Gamma(\beta+n)}{n! \Gamma(\gamma+n)} x^n. \quad (21)$$